3×3 Systems of Linear Equations and More Parabolas

There are many problems in math, science, and business that require more than two variables, for example, trying to find the path of a missile. To solve this problem, we will solve a system of *three equations* with *three variables*. [Note that a solution of such a system will consist of a set of three numbers that will satisfy all three of the equations.]

❑ *Example*

Solve the system:

$$\begin{aligned} \mathbf{4x - 2y + 3z} &= \mathbf{-14} \quad &\text{[Equ 1]} \\ \mathbf{6x + y - z} &= \mathbf{13} \quad &\text{[Equ 2]} \\ \mathbf{-x + 3y - 4z} &= \mathbf{24} \quad &\text{[Equ 3]} \end{aligned}$$

Start with Equ 1 and Equ 2 to eliminate the x:

$$\begin{aligned} 4x - 2y + 3z &= -14 \quad \text{(times 3)} &\Rightarrow\quad 12x - 6y + 9z &= -42 \\ 6x + y - z &= 13 \quad \text{(times } -2\text{)} &\Rightarrow\quad -12x - 2y + 2z &= -26 \end{aligned}$$

Add: $\quad -8y + 11z = -68 \quad$ [Equ 4]

Now use Equ 1 and Equ 3 to eliminate the x:

$$4x - 2y + 3z = -14 \quad \text{(leave it)} \quad \Rightarrow \quad 4x - 2y + 3z = -14$$

$$-x + 3y - 4z = 24 \quad \text{(times 4)} \quad \Rightarrow \quad -4x + 12y - 16z = 96$$

$$\text{Add:} \qquad 10y - 13z = 82 \quad \text{[Equ 5]}$$

Solve the 2×2 system consisting of Equ 4 and Equ 5:

$$\begin{array}{llll} \text{[Equ 4]} & -8y + 11z = -68 & \text{(times 5)} & \Rightarrow \; -40y + 55z = -340 \\ \text{[Equ 5]} & 10y - 13z = 82 & \text{(times 4)} & \Rightarrow \; 40y - 52z = 328 \end{array}$$

$$\text{Add:} \qquad 3z = -12$$

$$\Rightarrow \quad z = -4$$

We now calculate y using Equ 4 and our newly found value of z:

$$-8y + 11z = -68$$

$$\Rightarrow \quad -8y + 11(\underline{\mathbf{-4}}) = -68$$

$$\Rightarrow \quad -8y - 44 = -68$$

$$\Rightarrow \quad -8y = -24$$

$$\Rightarrow \quad y = 3$$

And lastly, we solve for x using Equ 1 and our values of y and z:

$$4x - 2y + 3z = -14$$

$$\Rightarrow \quad 4x - 2(\underline{\mathbf{3}}) + 3(\underline{\mathbf{-4}}) = -14$$

$$\Rightarrow \quad 4x - 6 - 12 = -14$$

$$\Rightarrow \quad 4x - 18 = -14$$

$$\Rightarrow \quad 4x = 4$$

$$\Rightarrow \quad x = 1$$

We conclude that the solution of the system of 3 equations in 3 variables is

$$\boxed{x = 1,\ y = 3,\ z = -4}$$

We could CHECK our solution by plugging the values of x, y, and z into ALL THREE of the *original* equations.

Homework

Solve each system of equations:

1. $$\begin{aligned} 2x - y + 3z &= 9 \\ x + 4y - z &= 6 \\ -3x + 2y + 2z &= 7 \end{aligned}$$

2. $$\begin{aligned} 3a - 4b + c &= 9 \\ -2a - 4b + 3c &= 19 \\ 5a + 3b - c &= -8 \end{aligned}$$

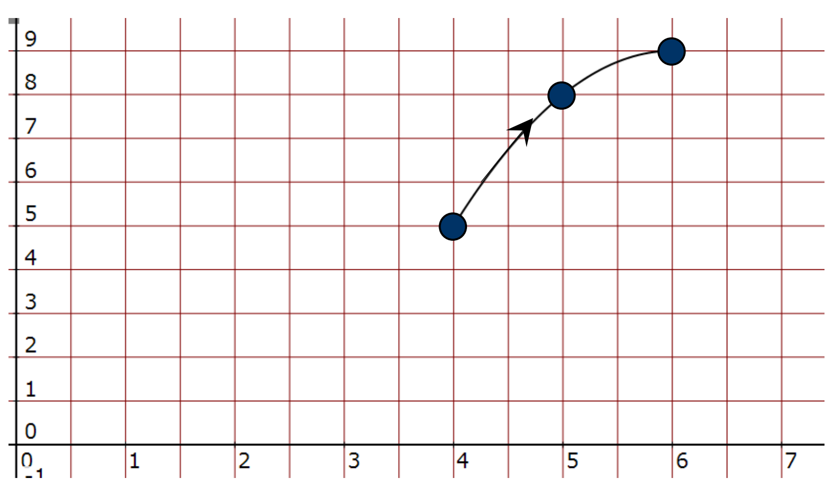
3. $$\begin{aligned} 2p + 3q - r &= 2 \\ -p - 3r + 2r &= 5 \\ 7p - 4p - 3r &= 23 \end{aligned}$$

4. $$\begin{aligned} 7x - 2y + 3z &= -12 \\ -5x + 7y - 4z &= 3 \\ 6x + 3y - 2z &= 6 \end{aligned}$$

❑ *APPLICATION*

A missile is fired from an unknown location on the *x*-axis of the radar grid. Its first three positions are reported as (4, 5), (5, 8), and (6, 9), at which time the radar jams. Assume that the missile follows a parabolic trajectory (path) traveling east. Calculate both the *x*-value (on the *x*-axis) from where the missile was fired and the *x*-value (on the *x*-axis) where the missile will hit the ground.

<u>Solution:</u> Here's a graphic of the missile's trajectory, as determined by the three reported positions:



Where was it launched, and where will it land?

Our goal is to "extrapolate" (extend) this curve to a full parabola formula, so we can deduce where the missile came from (pretty important) and where it's going to land (REALLY important).

Beginning with $y = ax^2 + bx + c$, we get the following three equations, obtained by plugging each of the points on the parabola into the generic parabola equation:

$$y = ax^2 + bx + c$$

(4, 5): $\mathbf{5} = a(\mathbf{4}^2) + b(\mathbf{4}) + c \;\Rightarrow\; 16a + 4b + c = 5$

(5, 8): $\mathbf{8} = a(\mathbf{5}^2) + b(\mathbf{5}) + c \;\Rightarrow\; 25a + 5b + c = 8$

(6, 9): $\mathbf{9} = a(\mathbf{6}^2) + b(\mathbf{6}) + c \;\Rightarrow\; 36a + 6b + c = 9$

We now have a system of three equations in three unknowns, which can be solved using the techniques of this chapter. When you (yes, you!) solve the system you will determine that $a = -1$, $b = 12$, and $c = -27$, from which we deduce that the parabola which describes the trajectory is

$$y = -x^2 + 12x - 27$$

But we're still not done. (By the way, we will not be blown to bits by the missile while we wade through all these calculations. A computer can solve the problem in a billionth of a second — but remember: someone has to program the computer.) To find the firing point and the landing point on the x-axis, we now calculate the two x-intercepts of the parabola, which we do by setting y to 0:

$$0 = -x^2 + 12x - 27$$

$$\Rightarrow \quad x^2 - 12x + 27 = 0$$

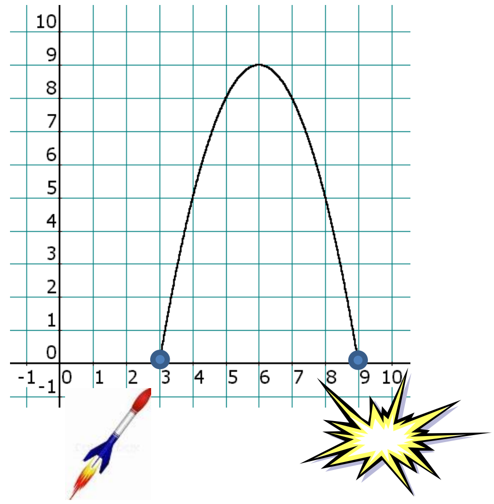
$$\Rightarrow \quad (x - 3)(x - 9) = 0$$

$$\Rightarrow \quad x - 3 = 0 \text{ or } x - 9 = 0$$

$\Rightarrow \quad x = 3$ or $x = 9 \;\Rightarrow\;$ The x-intercepts are **(3, 0)** and **(9, 0)**.

We're now ready to answer the question at hand:

> The missile was fired from $\boldsymbol{x = 3}$, and will land at $\boldsymbol{x = 9}$.



Homework

5. Solve the missile trajectory problem if the missile was spotted at the points (4, 14), (5, 18), (7, 20) before the radar failed.

6. Solve the missile trajectory problem if the missile was spotted at the points (7, 10), (8, 12) and (9, 12).

7. Solve the missile trajectory problem if the missile was spotted at the points (1, 0), (3, 8) and (5, 8).

8. Solve the missile trajectory problem if the missile was spotted at the points (3, 3), (4, 4) and (5, 3).

Solutions

1. $x = 1, y = 2, z = 3$

2. $a = 0, b = -1, c = 5$

3. $p = 4, q = -1, r = 3$

4. $x = 1/3, y = -10/3, z = -7$

5. Fired from $x = 2$; will land at $x = 11$

6. Fired from $x = 5$; will land at $x = 12$

7. Fired from $x = 1$; will land at $x = 7$

8. Fired from $x = 2$; will land at $x = 6$

"An education isn't how much
you have committed to memory,
or even how much you know.
It's being able to differentiate
between what you know and
what you don't."

Anatole France